Can Tho University Journal of Science
website: sj.ctu.edu.vn

DOI: 10.22144/ctu.jen.2018.015

A study of fixed points and Hopf bifurcation of Fitzhugh-Nagumo model

Phan Van Long Em*
An Giang university, Vietnam
**Correspondence: Phan Van Long Em (email: pvlem@agu.edu.vn)*

Article info.

Received 21 Feb 2017
Revised 21 Jun 2017
Accepted 30 Mar 2018

Keywords

FitzHugh-Nagumo model, fixed point, Hopf bifurcation, limit cycle

ABSTRACT

In this article, a class of FitzHugh-Nagumo model is studied. First, all necessary conditions for the parameters of system are found in order to have one stable fixed point which presents the resting state for this famous model. After that, using the Hopf's theorem proofs analytically the existence of a Hopf bifurcation, that is a critical point where a system's stability switches and a periodic solution arises. More precisely, it is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues cross the complex plane imaginary axis. Moreover, with the suitable assumptions for the dynamical system, a small-amplitude limit cycle branches from the fixed point.

Cited as: Em, P.V.L., 2018. A study of fixed points and Hopf bifurcation of Fitzhugh-Nagumo model. Can Tho University Journal of Science. 54(2): 112-121.

1 INTRODUCTION

In the beginning of 1960s, FitzHugh and Nagumo studied a model called FitzHugh-Nagumo model, to expose part of the inner working mechanism of the Hodgkin-Huxley equations, a famous model in study of neurophysiology since 1952. The FitzHugh-Nagumo model was introduced as a dimensional reduction of the well-known Hodgkin-Huxley model (Hodgkin and Huxley, 1952; Nagumo *et al*., 1962; Izhikevich, 2005; Ermentrout and Terman, 2009; Keener and Sney, 2009; Murray, 2010). It is constituted by two equations in two variables u and v. The first one is the fast variable called excitatory representing the transmembrane voltage. The second variable is the slow recovery variable describing the time dependence of several physical quantities, such as the electrical conductance of the ion currents across the membrane. The FitzHugh-Nagumo equations (FHN), using the notation in (Izhikevich and FitzHugh, 2006; Ambrosio, 2009; Ambrosio and Aziz-Alaoui, 2012; Ambrosio, 2012; Ambrosio and Aziz-Alaoui, 2013), are given by

$$\begin{cases} \dfrac{du}{dt}=\dot{u}=f(u,v)=u-\dfrac{u^3}{3}-v+I, \\ \dfrac{dv}{dt}=\dot{v}=g(u,v)=\dfrac{1}{\tau}(u+a-bv), \end{cases} \tag{1}$$

where u corresponds to the membrane potential, v corresponds to the slow flux ions through the membrane, I corresponds to the applied extern current, and $a,b,\tau\,(\tau>0)$ are parameters. Here, I,a,b,τ are real numbers.

The paper is organized as follows. In section 2, a study of fixed point is investigated and all necessary conditions for the parameters of FitzHugh-Nagumo model are found in order to have a stable focus. In section 3, the system undergoes supercritical Hopf bifurcation is shown. And finally, conclusions are drawn in Section 4.

2 A STUDY OF FIXED POINTS

Equilibria or stability are tools to study the dynamic of fixed points. In mathematics, a fixed point of a function is an element of the function's domain that is mapped to itself by the function. This paper focuses on the fixed points of the system (1) given by the resolution of the following system

$$\begin{cases} f(u,v)=0 \\ g(u,v)=0 \end{cases} \Leftrightarrow \begin{cases} u-\dfrac{u^3}{3}-v+I=0 \\ v=\dfrac{u+a}{b} \end{cases}$$

It implies that

$$u^3-\frac{3(b-1)}{b}u+\frac{3a}{b}-3I=0, \qquad (2)$$

where $b \neq 0$ (see $b=0$ in remark 2).

Let $p=-\frac{3(b-1)}{b}$ and $q=\frac{3a}{b}-3I$. The equation (2) can be written $u^3+pu+q=0$.

Let now $\Delta=4p^3+27q^2$.

If $\Delta>0$, then the equation (2) admits only one root and hence the system (1) admits a unique fixed point. Now, if $\Delta=0$, then the system (1) admits two fixed points, and finally if $\Delta<0$, the system (1) admits three fixed points (see Figure 1). This figure shows the numerical simulations obtained for two nullclines of the system (1) with $a=0.7, \tau=13$ and $I=0, \dot{u}=0$ in red and $\dot{v}=0$ in green. Figure 1(a) represents a unique fixed point of the system (1) for $b=0.8$; Figure 1(b) represents two fixed points for $b\simeq 2.3791$; and Figure 1(c) shows three fixed points for $b=3.5$.

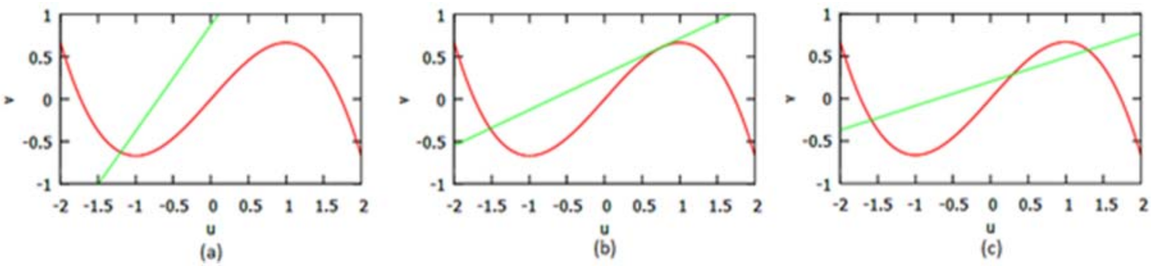

Fig. 1: Numerical simulations obtained for two nullclines of the system (1)

The Jacobian matrix of the system (1) is written as the following:

$$A(u)=\begin{pmatrix} \dfrac{\partial f(u,v)}{\partial u} & \dfrac{\partial f(u,v)}{\partial v} \\ \dfrac{\partial g(u,v)}{\partial u} & \dfrac{\partial g(u,v)}{\partial v} \end{pmatrix}=\begin{pmatrix} 1-u^2 & -1 \\ \dfrac{1}{\tau} & -\dfrac{b}{\tau} \end{pmatrix}.$$

Let (u^*,v^*) be one fixed point of (1), we have

$$Det(A(u^*)-\lambda\, \mathbf{I}_2)=\lambda^2-Tr(A(u^*))\lambda+Det(A(u^*)),$$

Where $Tr(A(u))=-u^2+1-\frac{b}{\tau}$ and

$$Det(A(u))=-\frac{b}{\tau}(1-u^2)+\frac{1}{\tau}=\frac{b}{\tau}u^2+\frac{1-b}{\tau}.$$

Note that, if $\tau>b$, then $Tr(A(u))$ admits two real roots given by

$$u_{Tr1}=\sqrt{1-\frac{b}{\tau}} \text{ and } u_{Tr2}=-\sqrt{1-\frac{b}{\tau}}.$$

The discriminant of $Det(A(u))$ is $-\frac{4b(1-b)}{\tau^2}$. Thus, if $b(1-b)<0 \Leftrightarrow b<0$ or $b>1$, then $Det(A(u))$ admits two real roots given by

$$u_{Det1}=\sqrt{1-\frac{1}{b}} \text{ and } u_{Det2}=-\sqrt{1-\frac{1}{b}}.$$

Here, the paper focuses on the case where the system has a unique fixed point. Moreover, note that if $b\in(0,1)$, then $p>0$, and hence $\Delta>0$. Thus, the value of b in $(0,1)$ is chosen.

Remark 1. When $b=1$, then $\Delta=0$ if $a=I$. This implies that the system (1) admits two fixed points. This case is not considered.

The type of fixed points can be resumed thank to the following tables.

In the case where $b<\tau$ and $0<b<1$.

Table 1: Stability of fixed point

u	$-\infty$	$-\sqrt{1-\frac{b}{\tau}}$		$\sqrt{1-\frac{b}{\tau}}$	$+\infty$
$Tr(A(u))$	$-$	0	$+$	0	$-$
$Det(A(u))$	$+$		$+$		$+$
Type of equilibrium	Stable focus Stable node		Unstable focus Unstable node		Stable focus Stable node

Remark 2. When $b=0$, a fixed point $E=(-a,-a+\frac{a^3}{3}+I)$ is obtained. The type of fixed points in this case is studied as the following table for $b=0$. In other words, the point E is stable if $|a|>1$, it is unstable if $|a|<1$, and it become a center if $|a|=1$.

In the case where $b\geq\tau$ and $0<b<1$.

Table 2: Stability of fixed point

u	$-\infty$	$+\infty$
$Tr(A(u))$	$-$	
$Det(A(u))$	$+$	
Type of equilibrium	Stable focus Stable node	

Look at Table 2, it is easy to see that the fixed point is always stable. It is not real for a neuron model. Remind that this work focuses on the context of slow - fast dynamics, so $\tau\gg 0$ (in particular, $\tau\gg b$). With $0<b<1$ and $\tau\gg 0$, a sufficient condition over the parameter a is found such that the stationary point is stable and stay at the left infinite branch of the cubic. Following Table 1, it is sufficient to have the stationary point (u^*,v^*) with $u^*\leq -1$ (since $-1<-\sqrt{1-\frac{b}{\tau}}$, this condition makes the fixed point stay at the left infinite branch of the cubic). Moreover, from the equation (2), we have

$$a=bu-\frac{b}{3}u^3-u+Ib.$$

By deriving the expression of a with respect to u, the above equation becomes

$$a'=b-bu^2-1<0,\forall b\in(0,1).$$

This implies the following table:

Table 3: Variation of the parameter a

u	$-\infty$		-1	$+\infty$
a'		$-$		
a	$+\infty$	$\searrow$	$1-\frac{2}{3}b+Ib$	

Table 3 shows that a sufficient condition is $1-\frac{2}{3}b+Ib<a$. To ensure the excitability character, $-2<u^*\leq -1$ is chosen. This implies that

$$1-\frac{2}{3}b+Ib<a<\frac{2}{3}b+2+Ib.$$

In particular, if $I=0$, it is easy to see that

$$1-\frac{2}{3}b<a<\frac{2}{3}b+2.$$

This condition permits to have a fixed point that is not so far from the local minimum of u-nullcline. Since, if the value of a is big enough, for example, $a>\frac{2}{3}b+2$, the fixed point will be far from the local minimum of u-nullcline. Therefore, the refractory period of the action potential will disappear (see Figure 2). The Figure 2(a) represents two nullclines of the system (1) with $a=3.5,b=0.8,\tau=13$ and $I=0,\dot{u}=0$ in red and $\dot{v}=0$ in green. The intersection point of two nullclines is the fixed point. The blue curve is obtained by drawing the asymptotic dynamic of one solution of the system starting from one initial condition. The Figure 2(b) shows the time series corresponding to (t,u)

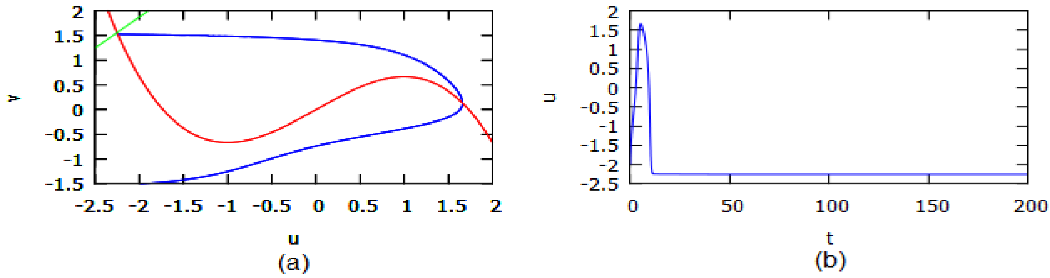

Fig. 2: Numerical results obtained for the system (1) with $a=3.5, b=0.8, \tau=13$ **,** $I=0$

Finally, the FitzHugh-Nagumo model of two equations is given by the following form:

$$\begin{cases} \dfrac{du}{dt}=\dot{u}=f(u,v)=u-\dfrac{u^3}{3}-v+I \\ \dfrac{dv}{dt}=\dot{v}=g(u,v)=\dfrac{1}{\tau}(u+a-bv) \end{cases} \qquad (3)$$

with

$$0<b<1, b\ll\tau, 1-\frac{2}{3}b+Ib<a<\frac{2}{3}b+2+Ib,$$

where u corresponds to the membrane potential, v corresponds to the slow flux ions through the membrane and I corresponds to the applied extern current.

In (3), we fix $a=0.7, b=0.8, \tau=13, I=0$, see (Izhikevich E. M., 2006) and Figure 3. We obtain

$$\begin{cases} \dfrac{du}{dt}=u-\dfrac{u^3}{3}-v \\ \dfrac{dv}{dt}=\dfrac{1}{13}(u+0.7-0.8v) \end{cases} \qquad (4)$$

The system (4) has one fixed point $B=(-1.1994,-062426)$. In Figure 3(a), we simulated two nullclines, $\dot{u}=0$ in red and $\dot{v}=0$ in green. The intersection point of these two nullclines is a fixed point B, and one orbit of (4) is represented in blue. At the point B, we get $Det(A)=0.1039$ and $Tr(A)=-0.5001$, hence $Tr(A)^2-4Det(A)<0$. Thus, B is a stable focus. The Figure 3(b) shows the time series corresponding to (t,u).

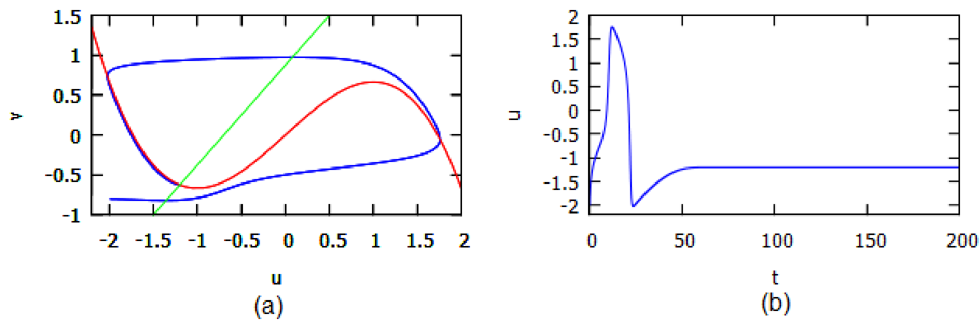

Fig. 3: Numerical results obtained for the system (4)

In particular, this system has the excitability property, thank to the following phenomenon: the initial condition $(u(0),v(0))$ is chosen on the left infinite branch of the cubic. Then,

if $(u(0),v(0))$ is such that the trajectory stays near enough from the local minimum

value of the cubic (see blue curve in Figure 4(a)), or it will not go to below of this value (see violet curve in Figure 4(a)), then the solution reaches closer to the stationary point quickly. More precisely, initially it is easy to see, $\dot{u}>0, \dot{v}<0$, and under the effect of the fast dynamic, the trajectory reaches closer to the left infinite branch. The solution tends to the stationary point under the effect of the slow dynamic and Figure 4(b) represents the time series corresponding to (t,u);

if $(u(0),v(0))$ is chosen such that the trajectory goes to below and far enough from the local minimum, then in this case the trajectory is not blocked any more by the left infinite branch and reaches to the right infinite branch under the effect of the fast dynamic. It takes up then this branch (since $\dot{v}>0$), under the effect of the slow dynamic, until v exceeds the local maximum, it then quickly joins the left branch, before finally reach slowly towards the equilibrium state. This system is thus said excitable, since when the solution is close to its equilibrium state, a disturbance can cause it to change greatly values before returning to its equilibrium state (see Figure 4(c)). This system thus provides a simple model of excitability that is observed in diverse cell (neurons, cardiomyocites, etc.). Figure 4(d) represents the time series corresponding to (t,u).

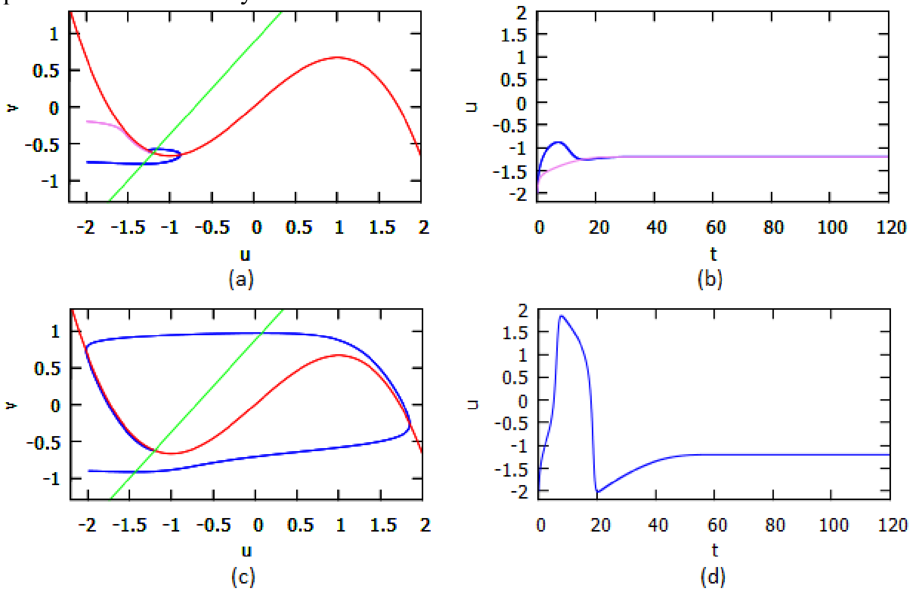

Fig. 4: Numerical solutions of the system (4)

3 EXISTENCE AND DIRECTION OF HOPF BIFURCATION

This section focuses on the existence and the direction of Hopf bifurcation, which corresponds to the passage of a fixed point to a limit cycle under the effect of variation of a parameter. Recall the Hopf's theorem (Dang-Vu Huyen, and Delcarte C., 2000; Corson N., 2009).

Theorem 1. Consider the system of two ordinary differential equations

$$\begin{cases} \dot{u}=f(u,v,a) \\ \dot{v}=g(u,v,a) \end{cases} \tag{5}$$

Let (u^*,v^*) a fixed point of the system (5) for all a. If the Jacobian matrix of the system (5) at (u^*,v^*) admits two conjugate complex eigenvalues, $\lambda_{1,2}(a)=\alpha(a)\pm iw(a)$ and there is a certain value $a=a_c$ such that $\alpha(a_c)=0, w(a_c)\neq 0$ *and* $\frac{\partial \alpha(a)}{\partial a}(a_c)\neq 0.$

Then, a Hopf bifurcation survive when the value of bifurcation parameter a passes by a_c and (u^*,v^*,a_c) is a point of Hopf bifurcation. Moreover, let c_1 in order that

$$c_1=\frac{1}{16w(a_c)}\left(-\frac{\partial^2 F}{\partial u^2}\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 F}{\partial u^2}\frac{\partial^2 F}{\partial u\partial v}-\frac{\partial^2 G}{\partial u^2}\frac{\partial^2 G}{\partial u\partial v}\right.$$
$$\left.-\frac{\partial^2 G}{\partial v^2}\frac{\partial^2 G}{\partial u\partial v}+\frac{\partial^2 F}{\partial v^2}\frac{\partial^2 F}{\partial u\partial v}+\frac{\partial^2 F}{\partial v^2}\frac{\partial^2 G}{\partial v^2}\right)+\left(\frac{\partial^3 F}{\partial u^3}+\frac{\partial^3 F}{\partial u\partial v^2}+\frac{\partial^3 G}{\partial u^2\partial v}+\frac{\partial^3 G}{\partial v^3}\right), \tag{6}$$

where F and G are given by the method of Hassard, Kazarinoff and Wan (Dang-Vu Huyen, and Delcarte C., 2000; Corson N., 2009).

We can distinguish different cases

Table 4: Stability of the fixed points according to Hopf bifurcation

		$c_1<0$	$c_1>0$
$\frac{\partial\alpha}{\partial a}(a_c)>0$	$a<a_c$	*stable equilibrium and no periodic orbit*	*stable equilibrium and unstable periodic orbit*
	$a>a_c$	*unstable equilibrium and stable periodic orbit*	*unstable equilibrium and no periodic orbit*
$\frac{\partial\alpha}{\partial a}(a_c)<0$	$a<a_c$	*unstable equilibrium and stable periodic orbit*	*unstable equilibrium and periodic orbit*
	$a>a_c$	*stable equilibrium and no periodic orbit*	*stable equilibrium and unstable periodic orbit*

Now this theorem is applied to the FitzHugh-Nagumo model in which I represents the bifurcation parameter

$$\begin{cases} \frac{du}{dt}=u-\frac{u^3}{3}-v+I \\ \frac{dv}{dt}=\frac{1}{13}(u+0.7-0.8v) \end{cases} \quad (7)$$

Let (u^*,v^*) a fixed point of the system (7). Let $u=u_1+u^*$ and $v=v_1+v^*$, then

$$\begin{cases} \dot{u}_1=f(u_1,v_1,I)=(u_1+u^*)-\frac{(u_1+u^*)^3}{3}-(v_1+v^*)+I \\ \dot{v}_1=g(u_1,v_1,I)=\frac{1}{13}\left[u_1+u^*+0.7-0.8(v_1+v^*)\right] \end{cases}$$

With a development of the functions f and g at the neighborhood of $(0,0,I)$, the above systems become

$$\begin{cases} \dot{u}_1=u_1\frac{\partial f}{\partial u_1}(0,0,I)+v_1\frac{\partial f}{\partial v_1}(0,0,I)+\widehat{F}(u_1,v_1,I) \\ \dot{v}_1=u_1\frac{\partial g}{\partial u_1}(0,0,I)+v_1\frac{\partial g}{\partial v_1}(0,0,I)+\widehat{G}(u_1,v_1,I) \end{cases}$$

where $\widehat{F}(u_1,v_1,I)$ and $\widehat{G}(u_1,v_1,I)$ are the nonlinear terms, then

$$\begin{cases} \dot{u}_1=(1-u^{*2})u_1-v_1+\widehat{F}(u_1,v_1,I) \\ \dot{v}_1=\frac{1}{13}u_1-\frac{4}{65}v_1+\widehat{G}(u_1,v_1,I) \end{cases}$$

with $\widehat{F}(u_1,v_1,I)=-\frac{u_1^3}{3}-\left(u_1^2-1\right)u^*-\frac{u^{*3}}{3}-v^*+I$ and $\widehat{G}(u_1,v_1,I)=\frac{1}{13}u^*+\frac{7}{130}-\frac{4}{65}v^*.$

Now, $(0,0,I)$ is a fixed point of the system. The Jacobian matrix is given by

$$A=\begin{pmatrix} 1-u^{*2} & -1 \\ \frac{1}{13} & -\frac{4}{65} \end{pmatrix}.$$

The characteristic polynomial

$$Det(A-\lambda\, \mathbf{I}_2)=\lambda^2+(u^{*2}-\frac{61}{65})\lambda+\frac{1}{65}+\frac{4}{65}u^{*2}.$$

Let $P(I)=-Tr(A)$ and $Q(I)=Det(A)$. We get

$$\lambda^2+P(I)\lambda+Q(I)=0.$$

Hence, the Jacobian matrix admits a pair of conjugate complex eigenvalues if $Det(A)>\frac{1}{4}Tr(A)^2$ and the above equation has the following roots

$\lambda_{1,2}=\alpha(I)\pm iw(I),$

with $\alpha(I)=-\frac{1}{2}u^{*2}+\frac{61}{130}$ and $w(I)=\sqrt{\frac{1}{65}+\frac{4}{45}u^{*2}-\alpha(I)^2}$. Recall that u^* is the solution of equation (2) which can be written $u^3+pu+q=0$ or $p=\frac{3}{4}$ and $q=\frac{21}{8}-3I$. This equation admits only one root, thank to the Cardan formulas that is given under the form

$$u^*(I)=m(I)+n(I),$$

with

$$\begin{cases} m(I)=\sqrt[3]{-\frac{21}{16}+\frac{3}{2}I+\frac{1}{2}\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I\right)^2}} \\ n(I)=\sqrt[3]{-\frac{21}{16}+\frac{3}{2}I-\frac{1}{2}\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I\right)^2}} \end{cases}$$

Moreover, the value I_c of I, for which the real part of these eigenvalues is null, is given by the equations $P(I_c)=0$ and $Q(I_c)>0$, then

$$u^*(I_c)^2-\frac{61}{65}=0\Rightarrow u^*(I_c)=\pm\sqrt{\frac{61}{65}}.$$

First, the value $u^*(I_C)=-\sqrt{\frac{61}{65}}$ is considered. Thank to the equation (2), it is easy to obtain

$$I_C=\frac{7}{8}-\frac{439}{780}\sqrt{\frac{61}{65}}.$$

Moreover,

$$\frac{\partial\alpha}{\partial I}(I_C)=-u^*(I_C)\left[\frac{dm(I)}{dI}(I_C)+\frac{dn(I)}{dI}(I_C)\right]$$

$$=-\frac{1}{3}u^*(I_C)\left[\frac{1}{m(I_C)^{2/3}}\left(\frac{3}{2}-(\frac{63}{16}-\frac{9}{2}I_C)\frac{1}{\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I_C\right)^2}}\right)+\frac{1}{n(I_C)^{2/3}}\left(\frac{3}{2}+(\frac{63}{16}-\frac{9}{2}I_C)\frac{1}{\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I_C\right)^2}}\right)\right]$$

$$\approx 0.8788\neq 0.$$

Thus, $\alpha(I_C)=0, w(I_C)\neq 0$ and $\frac{\partial\alpha(I)}{\partial I}(I_C)\neq 0$, then I_C is a bifurcation Hopf value of the parameter I.

In the following, the direction and the stability of Hopf bifurcation are investigated. To do this, let's determine an eigenvector V_1 associated with the eigenvalue λ_1, obtained by resolving the system

$$(A-\lambda_1 I_2)\begin{pmatrix}u\\v\end{pmatrix}=0\Rightarrow\begin{cases}(1-u^{*2}+iw_0)u-v=0\\ \frac{1}{13}u-\left(\frac{4}{65}+iw_0\right)v=0\end{cases}$$

where $w_0=w(I_C)$. A solution of this system is an eigenvector associated with λ_1 given by

$$V_1=\begin{pmatrix}1\\1-u^{*2}-iw_0\end{pmatrix}.$$

The base change matrix is given by

$$P=\left(\mathrm{Re}(V_1)\quad \mathrm{Im}(V_1)\right)=\begin{pmatrix}1 & 0\\ 1-u^{*2} & w_0\end{pmatrix}.$$

Then

$$P^{-1}=\frac{1}{w_0}\begin{pmatrix}w_0 & 0\\ u^{*2}-1 & 1\end{pmatrix}.$$

Now let the variable change

$$\begin{pmatrix}u_1\\v_1\end{pmatrix}=P\begin{pmatrix}u_2\\v_2\end{pmatrix}\Rightarrow\begin{pmatrix}u_2\\v_2\end{pmatrix}=P^{-1}\begin{pmatrix}u_1\\v_1\end{pmatrix}.$$

Hence

$$\begin{pmatrix}\dot{u}_2\\ \dot{v}_2\end{pmatrix}=P^{-1}\begin{pmatrix}\dot{u}_1\\ \dot{v}_1\end{pmatrix}=P^{-1}AP\begin{pmatrix}u_2\\v_2\end{pmatrix}+P^{-1}\begin{pmatrix}\widehat{F}(u_2,v_2,I)\\ \widehat{G}(u_2,v_2,I)\end{pmatrix}.$$

Let $A'(I)=P^{-1}AP=\begin{pmatrix}\alpha(I) & -w(I)\\ w(I) & \alpha(I)\end{pmatrix}$. Then, for $I=I_C$, it implies that

$$A'(I_C)=\begin{pmatrix}0 & -w(I_C)\\ w(I_C) & 0\end{pmatrix}\Rightarrow\begin{cases}\dot{u}_2=-w(I_C)v_2+F(u_2,v_2,I)\\ \dot{v}_2=w(I_C)u_2+G(u_2,v_2,I)\end{cases}$$

with

$$\begin{pmatrix}F(u_2,v_2,I_C)\\ G(u_2,v_2,I_C)\end{pmatrix}=P^{-1}\begin{pmatrix}\widehat{F}(u_2,v_2,I_C)\\ \widehat{G}(u_2,v_2,I_C)\end{pmatrix}.$$

Then

$$\begin{cases}F(u_2,v_2,I_C)=-\frac{1}{3}u_2^3-(u_2^2-1)u^*-\frac{1}{3}u^{*3}-v^*+I_C=M\\ G(u_2,v_2,I_C)=M\left(\frac{u^{*2}-1}{w_0}\right)+\frac{1}{w_0}\left(\frac{u^*}{13}+\frac{7}{130}-\frac{4}{65}v^*\right)\end{cases}$$

Let c_1 be given by the equation (6). The functions F and G depend only on u_2, the coefficient c_1 is given by

$$c_1=-\frac{1}{16w(I_C)}\frac{\partial^2F}{\partial u_2^2}(0,0,I_C)\frac{\partial^2G}{\partial u_2^2}(0,0,I_C)+\frac{\partial^3F}{\partial u_2^3}(0,0,I_C).$$

At the point $(u_2,v_2)=(0,0)$ and for $I=I_c$, it implies that $w(I_c)=\sqrt{\frac{1}{65}+\frac{4}{65}u^{*2}}$, and

$$c_1=-\frac{u^{*2}(u^{*2}-1)}{4w_0^2}-2=-\frac{557}{309}<0,$$

with $u^*=-\sqrt{\frac{61}{65}}$. The theorem 1 permits to deduce the direction and the stability of Hopf bifurcation from the signs of $\frac{\partial\alpha}{\partial I}(I_c)$ and c_1. Following Table 4, since $\frac{\partial\alpha}{\partial I}(I_c)>0$ and $c_1<0$, it is easy to see that (u^*,v^*,I_c) is a supercritical Hopf bifurcation point.

Moreover, for $I>I_c$, the fixed point is unstable with a stable periodic orbit, while for $I<I_c$, the fixed point is stable and there is not the periodic orbit (see Figure 5). Figure 5(a) shows the phase portrait in the plane (u,v) of the system (7) with $I=0.3$, and a focus stable for a value $I=0.3<I_c$. Figure 5(b) represents the time series corresponding to (t,u). Figure 5(c) shows the phase portrait in the plane (u,v) of the system (7) with $I=0.4$, and a stable limit cycle for a value $I=0.4>I_c$. Figure 5(d) represents the time series corresponding to (t,u).

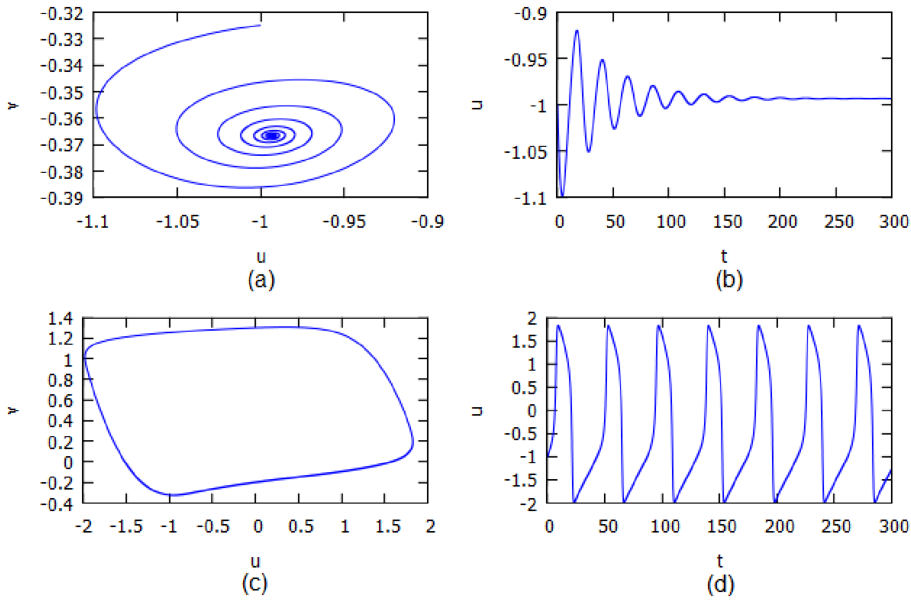

Fig. 5: (a) Phase portrait in the plane (u,v) of the system (7) with $I=0.3$, shows a focus stable for a value $I=0.3<I_c$. (b) Time series corresponding to (t,u). (c) Phase portrait in the plane (u,v) of the system (7) with $I=0.4$, shows a stable limit cycle for a value $I=0.4>I_c$. (d) Time series corresponding to (t,u)

Similarly, let's repeat the previous process for $u^*(I_c^{'})=\sqrt{\frac{61}{65}}$. Then the associated value of I can be also found as the following

$$I_c^{'}=\frac{439}{780}\sqrt{\frac{61}{65}}+\frac{7}{8}$$

and

$$\frac{\partial \alpha}{\partial I}(I_c^{'}) = -u*'(I_c^{'})\left[\frac{dm(I)}{dI}(I_c^{'})+\frac{dn(I)}{dI}(I_c^{'})\right]$$

$$=-\frac{1}{3}4u*'(I_c^{'})\left[\frac{1}{m(I_c^{'})^{2/3}}\left(\frac{3}{2}-(\frac{63}{16}-\frac{9}{2}I_c^{'})\frac{1}{\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I_c^{'}\right)^2}}\right)\right.$$

$$\left.+\frac{1}{n(I_c^{'})^{2/3}}\left(\frac{3}{2}+(\frac{63}{16}-\frac{9}{2}I_c^{'})\frac{1}{\sqrt{\frac{1}{16}+\left(\frac{21}{8}-3I_c^{'}\right)^2}}\right)\right]$$

$$\approx -0.8788 \neq 0.$$

Thus, $\alpha(I_c^{'})=0, w(I_c^{'})\neq 0$ and $\frac{\partial \alpha(I)}{\partial I}(I_c^{'})\neq 0$, then $I_c^{'}$ is a Hopf bifurcation value of the parameter I and then $c_1=-\frac{557}{309}<0$.

Now $\frac{\partial \alpha}{\partial I}(I_c^{'})<0$ and $c_1<0$. Following Table 4, $(u^*,v^*,I_c^{'})$ is a supercritical Hopf bifurcation point.

Moreover, for $I<I_c^{'}$, the fixed point is unstable with a stable periodic orbit, while for $I>I_c^{'}$, the fixed point is stable and there is not the periodic orbit (see Figure 6). Figure 6(a) shows the phase portrait in the plane (u,v) of the system (7) with $I=1.4$, and a stable limit cycle for a value $I=1.4<I_c^{'}$. Figure 6(b) represents the time series corresponding to (t,u). Figure 6(c) shows the phase portrait in the plane (u,v) of the system (7) with $I=1.5$, and a stable focus for a value $I=1.5>I_c^{'}$. Figure 6(d) represents the time series corresponding to (t,u).

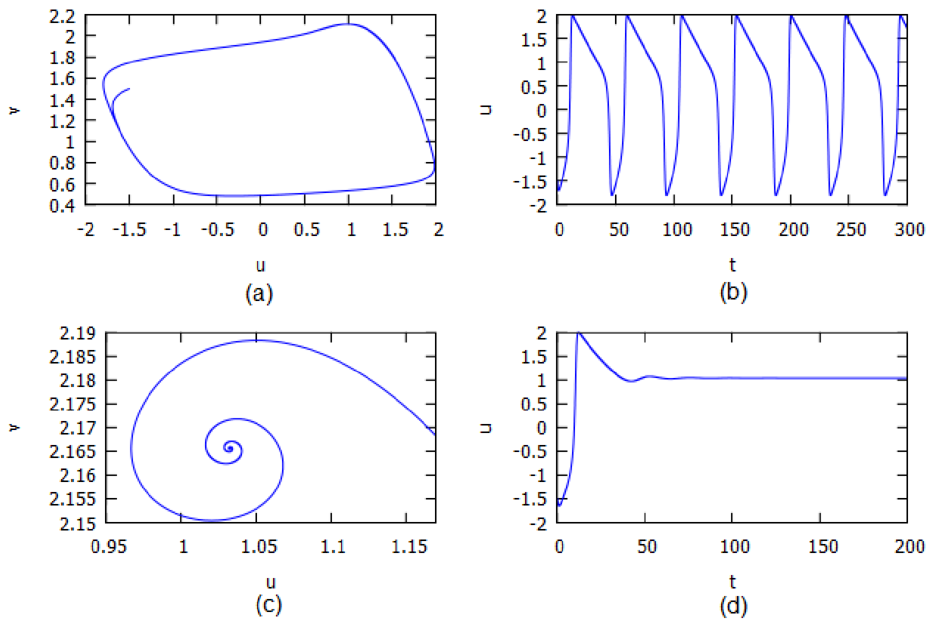

Fig. 6: (a) Phase portrait in the plane (u,v) of the system (7) with $I=1.4$, shows a stable limit cycle for a value $I=1.4<I_c^{'}$. (b) Time series corresponding to (t,u). (c) Phase portrait in the plane (u,v) of the system (7) with $I=1.5$, shows a stable focus for a value $I=1.5>I_c^{'}$. (d) Time series corresponding to (t,u)

In Figure 7, a bifurcation diagram in function of I is simulated in the plane (I,u).

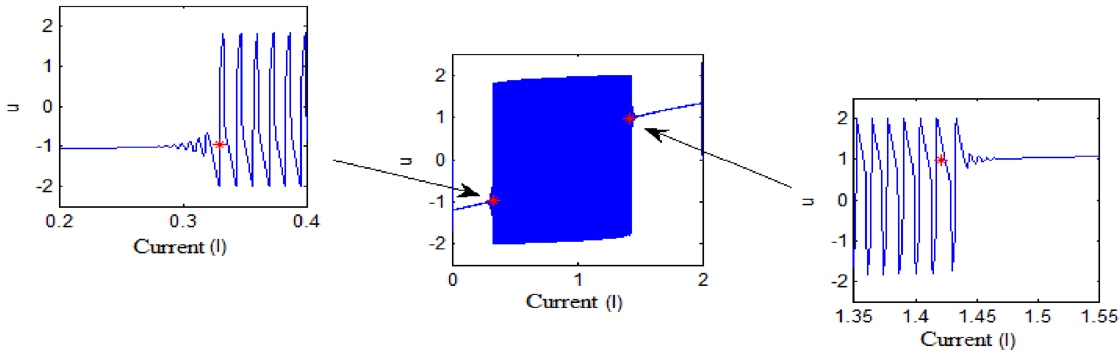

Fig. 7: Bifurcation diagram in function of I in the plane (I,u)

Figure 7 shows the adhesion orbits from different values of I. This illustrates the supercritical Hopf bifurcation at the bifurcation point obtained analytically, and the appearance of an attractive limit cycle. There is a bifurcation or a stability change when I acrosses the values I_C and $I_C^{'}$ (two red stars in Figure 7). If I is between these two values, the system turns around a limit cycle asymptotically while if I is outside of the interval $\left[I_C;I_C^{'}\right]$, then the system converges to a stable fixed point.

4 CONCLUSION

This work showed the necessary conditions for the parameters of FitzHugh-Nagumo model such that there exists only a stable fixed point. It represents the resting state in this system. The applied extern current is chosen like a bifurcation parameter, and when it crosses through the bifurcations values, then the equilibrium point loses its stability and becomes a limit cycle that implies the existence of a Hopf bifurcation. In this paper, the FitzHugh-Nagumo model has two bifurcation values where there exists the supercritical Hopf bifurcation and they are illustrated by a bifurcation diagram. The future work will be studied about the chaos properties in the FitzHugh-Nagumo by adding some perturbation parameters.

REFERENCES

Ambrosio, B., and Aziz-Alaoui, M.A., 2012. Synchronization and control of coupled reaction diffusion systems of the FitzHugh-Nagumo-type. Computers and Mathematics with Applications. 64: 934-943.

Ambrosio, B., 2009. Propagation of waves in an excitable zone: numerical simulations and analytical approach. PhD thesis. The university of Pierre and Marie Curie-Paris 6. Paris, France (in french).

Ambrosio, B., and Aziz-Alaoui, M.A., 2012. Synchronization in a network of coupled reaction diffusion systems of the FitzHugh-Nagumo-type. Actes du colloque EDP-Normandie, Le Havre, 119-131 (in French).

Ambrosio, B., and Aziz-Alaoui, M.A., 2013. Synchronization and control of a network of coupled reaction-diffusion systems of generalized FitzHugh-Nagumo type. ESAIM: Proceedings. 39: 15-24.

Aziz-Alaoui, M.A., 2006. Synchronization of Chaos. Encyclopedia of Mathematical Physics. Elsevier. 5: 213-226.

Corson, N., 2009. Dynamics of a neural model, synchronization and complexity. PhD thesis. The university of Le Havre. Le Havre, France (in french).

Dang-Vu Huyen, and Delcarte, C., 2000. Bifurcations and Chaos, an introduction to dynamics contemporary with programs in Pascal, Fortan et Mathematica. Eds Elipses, Université – Mécanique (in french).

Ermentrout, G. B., Terman, D. H., 2009. Mathematical Foundations of Neurosciences. Springer.

Hodgkin, A.L., and Huxley, A. F., 1952. A quantitative description of membrane current and its application to conduction and excitation in nerve. J. Physiol. 117: 500-544.

Izhikevich, E. M., 2007. Dynamical Systems in Neuroscience. The MIT Press.

Izhikevich, E. M., and FitzHugh, R., 2006. FitzHugh-Nagumo model. Scholarpedia, 1(9):13-49.

Keener, J. P., and Sneyd, J., 2009. Mathematical Physiology. Springer.

Murray, J. D., 2010. Mathematical Biology. Springer.

Nagumo, J., Arimoto, S., and Yoshizawa, S., 1962. An active pulse transmission line simulating nerve axon. Proc. IRE. 50: 2061-2070.