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Solving a mathematical problem in different ways: A case of calculating the distance from a point to a plane

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ABSTRACT

Problem solving helps students to consolidate their knowledge, skills and develop their thinking. In particular, solving a mathematical problem in different ways promotes more of this meaning because to do this, students must mobilize a lot of knowledge, skills and various kinds of thinking. Therefore, in learning mathematics, to solve a problem in a number of ways is one of the ways to develop students' creative thinking. A sample consisted of 138 students in Can Tho city and Hau Giang province and they solved the problem of calculating the distance from a point to a plane. The results showed that a large number of students only solved it in one way and only 32 students addressed it in two ways. Moreover, when solving the problem, students made many errors for various reasons.

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1 INTRODUCTION

1.1 Solving a problem in different ways in mathematics and its meanings

Problem solving is a process of exploring, thinking, and applying the knowledge that students already know. For a variety of reasons, students may think and seek a solution for a long time, then find another way to solve the problem. At the same time, some students discover solutions very fast and find a variety of solutions. This difference comes from many various reasons.

Polya (1973) identified four basic principles of problem solving including understanding the problem, devising a plan, carrying out the plan, and looking back. In the second principle, the author mentioned that there were many reasonable ways to solve problems. In addition, Polya's famous statement about the value of solving a problem in many ways is "It is better to solve one problem five different ways, than to solve five problems one way".

The Vietnamese author, Pham Dinh Thuc (2009) pointed out the benefits of solving problems in a variety of ways. Solving a mathematical problem in a variety of ways will assist students in developing problem solving skills, the reasoning ability when solving a problem in different ways in different situations. Thus, it helps students find good and short solutions to the problem. From there, it develops students' perseverance, creativity in learning and practicing their own problem solving and can be applied to the handling of life situations in the most optimal way. Solving mathematical problems in a variety of ways will help teachers find a good teaching direction for each student, develop problem-solving skills and be able to cover the entire curriculum. From there, they have a better and more effective way of teaching. In teaching mathematics, teachers need to ask students to have a habit of thinking about looking for different solutions to a problem. By that, it will help students see that the mathematics is not monotonous, rigid, but very lively and interesting; therefore, the students will be fascinated and interested in the subject, then begin to explore the subject. This will support students in learning mathematics better and applying the subject in a flexible and effective way.

Bingolbali (2011) said that solving problems in different ways was necessary for mathematics learning and teaching. For this reason, 500 teachers were asked to answer questions regarding students' different solutions. Survey results showed that they did not value the different solutions and found it difficult to grade them.

Loc *et al*. (2016) asked 31 twelfth grade students to calculate the integral $\int_0^{\frac{\pi}{2}} \sin(\frac{\pi}{4} - x)dx$ in different ways. Surprisingly, the students were very motivated to come up with different solutions because they knew how to flexibly apply knowledge to solve the problem. However, some students still committed a number of significant errors, such as lack of upper and lower values and incorrect primitive formulas. Similarly, Loc and Tong (2016) explored primary school students' problem - solving ability through addressing the problem "Compare two fractions $\frac{4}{5}$ and $\frac{3}{2}$" in a variety of ways. It was revealed that the participants had the ability of problem solving to discover many solutions for the given problem.

1.2 The ways of calculating the distance from a point to a plane in textbooks in Vietnam

The distance from a point to a plane is represented in the 11th grade mathematics textbook (Tran Van Hao, 2015a) in a general problem "Given point *O* and plane (α). Let *H* be the perpendicular projection of *O* on the plane (α). Now the distance between two points *O* and *H* is called the distance from the point *O* to the plane (α), denoted by $d(O;(\alpha))$. From here, there is a way of calculating the distance from one point to the plane as follows: $MH \perp (\alpha) \Rightarrow d\left(M;(\alpha)\right) = MH$.

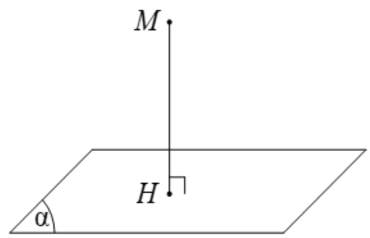

Fig. 1: The way of calculating the distance from a point to a plane related to finding the perpendicular projection of the given point on the plane

In grade 12, students learn more about the volume of a solid pyramid with the area of the base *B* and the height *h* is $V = \frac{1}{3}B.h$. For example, to compute the volume of solid tetrahedron *D.ABC*, students can use the formula $V_{D.ABC} = \frac{1}{3}S_{\Delta ABC}.DH$. From this formula, they infer $d(D;(ABC)) = DH = \frac{3V_{D.ABC}}{S_{\Delta ABC}}$. Moreover, in the 12th grade mathematical program, students learn one more way of calculating the distance from a point to a plane through the following theorem.

In the space *Oxyz*, given the plane (α) with the equation $Ax + By + Cz + D = 0$ and the point $M_0(x_0;y_0;z_0)$. The distance from the point M_0 to the plane (α), denoted by $d(M_0;(\alpha))$, is calculated by the formula: $d(M_0;(\alpha)) = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$ (Tran Van Hao, 2015b).

From the three basic methods (classical geometry method, volume method and coordinate method), the students can vary for different ways depending on various mathematical problems. The problem is whether the students are successful in using the three methods to solve a distance problem.

1.3 Research objectives

The goal of the study was to investigate the ability of 12th graders to solve the problem of calculating the distance from a point to a plane in different ways. Specifically, this study sought answers to the following questions:

Q1: What ways do students use to solve the problem? Which way is prioritized?

Q2: Is there a difference between the students in Can Tho and in Hau Giang in solving the problem in different ways?

Q3: Is there a difference between boys and girls in solving the problem in different ways?

Q4: What errors in solving the problem can students make? What may be the reasons?

2 METHODS

2.1 Participants

The participants were 138 grade-12 students from Thuan Hung High School, Can Tho and Le Quy Don High School, Hau Giang. Grades 12A1, 12A2 belonged to Thuan Hung High and grades 12A, 12B were chosen from Le Quy Don High School. The facilities, teaching conditions were relatively full, most students of these classes had a good learning aptitude.

At the time of the survey, students completed the chapter "The Method of Coordinates in Space" and had an understanding of the coordinate method. Theoretically, students had enough knowledge to solve the survey problem in all three methods predicted: classical geometry method, volume method and coordinate method, here the three methods were divided mostly, but different solution ways could appear.

2.2 Instrument and procedure

Students were asked to solve the following problem in different ways within 45 minutes.

Given pyramid *S.ABCD* whose base is square *ABCD* with edge *a*, with *SA* perpendicular to the plane *(ABCD)* and the edge $SA = a\sqrt{2}$. Let *O* be the center of square *ABCD*. Calculate the distance from the point *O* to the plane *(SCD)*.

Possible solutions

• S1: Using classical geometry method

S1.1: Find the projection of O on the plane (SCD)

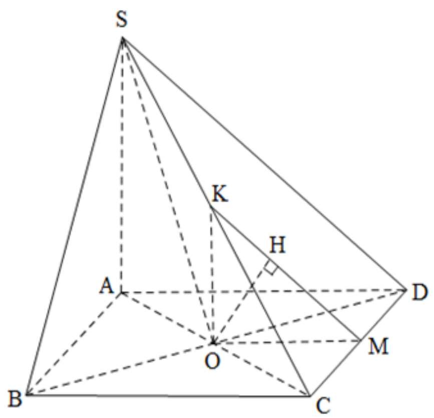

Fig. 2: The figure of the problem related to S1.1

Let *M, K* be the midpoints of two edges *CD* and *SC*, respectively.

In the triangle *SAC*, we have the average line *OK*, so $OK // SA,\ OK = \frac{1}{2}SA = \frac{a\sqrt{2}}{2}$, therefore $OK \perp (ABCD)$, so $OK \perp CD$.

We have $OM \perp CD$. We infer $CD \perp (OMK)$. Draw $OH \perp KM$; $H \in KM$.

But, $CD \perp (OMK)$, so $OH \perp CD$. Hence, $OH \perp (SCD)$, therefore $d(O;(SCD)) = OH$.

In the right triangle OKM, we have $\frac{1}{OH^2} = \frac{1}{OM^2} + \frac{1}{OK^2}$, so $OH = \frac{a\sqrt{6}}{6}$

S1.2: Based on the distance from A to plane (SCD)

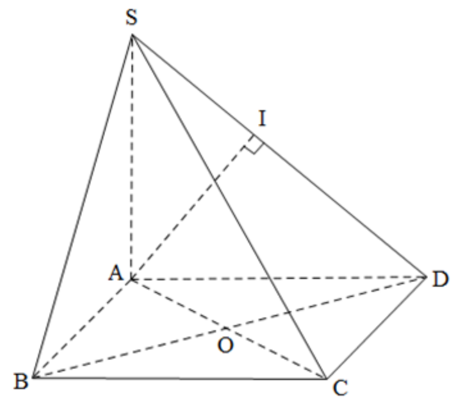

Fig. 3: The figure of the problem related to S1.2

Draw $AI \perp SD$; $I \in SD$ We have $\begin{cases} CD \perp SA \\ CD \perp AD \end{cases}$, so $CD \perp (SAD)$, therefore $CD \perp SI$. Also, $AI \perp SD$, so $AI \perp (SCD)$ and $d(A;(SCD)) = AI$.

In the right triangle *SAD*, we have $\frac{1}{AI^2} = \frac{1}{SA^2} + \frac{1}{AD^2} \Rightarrow AI = \frac{a\sqrt{6}}{3}$.

On the other hand, $\frac{d(O;(SCD))}{d(A;(SCD))} = \frac{OC}{AC} = \frac{1}{2}$, so $d(O;(SCD)) = \frac{1}{2}AI = \frac{a\sqrt{6}}{6}$.

S1.3: Based on the distance from P to the plane (SCD) (P is the midpoint of AD)

Let *P* be the midpoint of *AD.*

We have $OP // CD \Rightarrow OP // (SCD)$, so $d(O;(SCD)) = d(P;(SCD))$.

Draw $PN \perp SD$, $N \in SD$.

On the other hand, $CD \perp (SAD)$, therefore $CD \perp PN$. We infer $PN \perp (SCD)$.

In the right triangle *SAD*, we have $SC = \sqrt{SA^2 + AD^2} = \sqrt{(a\sqrt{2})^2 + (a)^2} = a\sqrt{3}$.

We have $\frac{PN}{SA} = \frac{PD}{SD}$, so $PN = \frac{SA.PD}{SD} = \frac{a\sqrt{2}.\frac{a}{2}}{a\sqrt{3}} = \frac{a\sqrt{6}}{6}$.

So $d(O;(SCD)) = d(P;(SCD)) = PN = \frac{a\sqrt{6}}{6}$.

• S2: Using the volume method

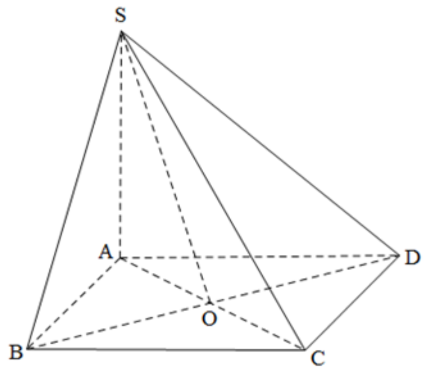

Fig. 4: The figure of the problem related to S2

We have $V_{S.OCD} = \frac{1}{3} S_{\Delta SCD}.SA = \frac{1}{3}.\frac{1}{2}(\frac{a\sqrt{2}}{2})^2.a\sqrt{2}$.

In the right triangle *SAC*, we have $SC = \sqrt{SA^2 + AC^2} = \sqrt{(a\sqrt{2})^2 + (a\sqrt{2})^2} = 2a$.

Similarly, $SD = a\sqrt{3}$. We have $SC^2 = SD^2 + CD^2$, so ΔSCD is right at *D*.

We have $S_{SCD} = \frac{1}{2} SD.CD = \frac{1}{2} a\sqrt{3}.a = \frac{a^2\sqrt{3}}{2}$.

$V_{S.OCD} = \frac{1}{3} S_{\Delta SCD}.d(O;(SCD))$, from there $d(O;(SCD)) = \frac{3V_{S.OCD}}{S_{\Delta SCD}} = \frac{a\sqrt{6}}{6}$.

S3: Using the coordinate method

S3.1: Coordinate with A as the origin

The problem can be represented by a coordinate system *Oxyz* as follows:

$A(0;0;0); S(0;0;a\sqrt{2}); B(a;0;0);$

$C(a;a;0); D(0;a;0); O\left(\frac{a}{2};\frac{a}{2};0\right);$

$\overrightarrow{SC} = (a;a;-a\sqrt{2}); \overrightarrow{SD} = (0;a;-a\sqrt{2})$;

$[\overrightarrow{SC};\overrightarrow{SD}] = (0;a^2\sqrt{2};a^2) = a^2(0;\sqrt{2};1)$.

The plane (*SCD*) has the normal vector $\vec{n} = (0;\sqrt{2};1)$ and passes through *D*, so it has the equation: $\sqrt{2}y + z - a\sqrt{2} = 0$. Hence:

$$d(O;(SCD)) = \frac{\left|\sqrt{2}.\frac{a}{2} - a\sqrt{2}\right|}{\sqrt{(\sqrt{2})^2 + 1^2}} = \frac{a\sqrt{6}}{6}.$$

S3.2: Coordinate with O as the origin

The problem can be represented by a coordinate system *Oxyz* as follows:

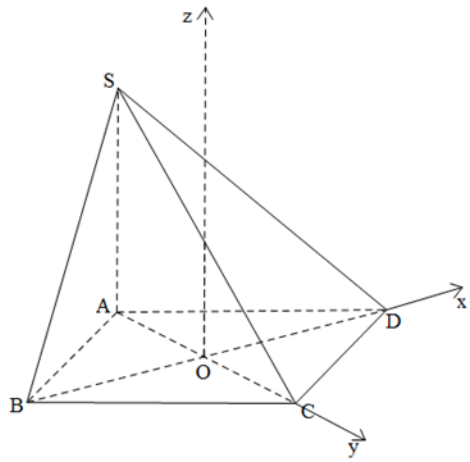

Fig. 5: The figure of the problem related to S3.2

$O(0;0;0); B\left(-\frac{a\sqrt{2}}{2};0;0\right); C\left(0;\frac{a\sqrt{2}}{2};0\right);$

$D\left(\frac{a\sqrt{2}}{2};0;0\right); A\left(0;-\frac{a\sqrt{2}}{2};0\right); S\left(0;-\frac{a\sqrt{2}}{2};a\sqrt{2}\right)$

$\overrightarrow{SD} = \left(\frac{a\sqrt{2}}{2};\frac{a\sqrt{2}}{2};-a\sqrt{2}\right); \overrightarrow{SC} = (0;a\sqrt{2};-a\sqrt{2})$;

$[\overrightarrow{SC};\overrightarrow{SD}] = (-a^2;-a^2;-a^2) = -a^2(1;1;1)$.

The plane (*SCD*) has the normal vector $\vec{n} = (1;1;1)$ and passes through *D*, so it has the equation: $x + y + z - \frac{a\sqrt{2}}{2} = 0$. Hence,

$$d(O;(SCD)) = \frac{\left|-\frac{a\sqrt{2}}{2}\right|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{a\sqrt{6}}{6}.$$

3 RESULTS AND DISCUSSION

The problem was not too difficult for students with good learning aptitude, so in 45 minutes the students were able to have enough time to think and present their solutions in many ways as possible. The results were as follows:

Table 1: Statistics of students in the number of ways they did

	S1.1	S1.2	S1.3	S2	S3.1	S3.2
The number of students	16	44	5	44	71	2
The number of students with correct solution	6	42	5	41	65	2

The majority of students only had a solution way, specifically, there were 98 students solving problems in only one way, accounting for 71.01% of the 138 students surveyed. Furthermore, 32 students (23.19%) solved the problem in two ways, and these two ways students mainly used were S1.2 and S3.1. In parallel, five students addressed the problem in three correct ways, including four students in 12A1 class –Thuan Hung High School class and one student in class 12A2 of this school; in particular, a student with four correct solution ways still belonged to class 12A1. This proved that the creative thinking of students was not high.

Based on the results of Table 2, it is found that S3.1 was the most preferred method for students, accounting for 51.4% of the total students surveyed. S3.1 was the solution way that most students chose to solve the distance problem because they had just completed the coordinate method and applied this method to classroom assignments and homework assignments. The majority of students chose the right coordinate system of *Oxyz*, wrote the equation of the plane and applied the formula to calculate the distance from a point to a plane. Of these 71 students, 65 students made up 91.5% of the total number of students with correct solution. Six students had inaccurate explanations, of which three students failed to write the plane equations, and three students simplified the plane equations incorrectly, leading to the wrong result of the distance.

3.1 To Q1

From the survey results summarized in Table 1 and Table 2, the answer to Q1 was raised as follows:

3.2 To Q2

From the results of the survey summarized in Table 3 and Table 4, the answer to Q2 was mentioned as follows:

Table 3: Student statistics with the correct answers in each solution in Can Tho

	S1.1	S1.2	S1.3	S2	S3.1	S3.2
The number of students	11	42	5	22	30	0
The number of students with the correct solution	3	41	5	20	28	0

Table 4: Student statistics with the correct answers in each solution in Hau Giang

	S1.1	S1.2	S1.3	S2	S3.1	S3.2
The number of students	5	2	0	22	41	2
The number of students with the correct solution	3	1	0	21	37	2

For S1.2, students in Can Tho were better than those in Hau Giang. The rate of students in Can Tho with the correct answers was 97.6%, while this percent in Hau Giang was about 50%. In contrast, the students in Hau Giang followed the S1 way more accurately than the students in Can Tho did, with 60% and 27.3% respectively. Also, for S3.1, the percentage of students solving the problem correctly in two places was almost the same. There was no significant difference in the proportion of students with the right answers in Can Tho city and Hau Giang province when they chose S2. Compared to the number of solution ways, the results were the same when students in both places found five solutions.

S3.2 was the solution way that students in Can Tho did not discover, while the students in Hau Giang did not find out S1.3. S1.2 and S3.1 were the two ways that most students in Can Tho selected, with 41 and 28 selections respectively. Meanwhile, S2 and S3.1 were chosen by many students in Hau Giang, with 22 and 41 choices respectively.

In general, these results partly reflected reality in two classes of two schools in two localities, suitable to the learning aptitude and learning conditions of students in two provinces. Withdrawing, the two groups of these students did not have much difference in both problem-solving ability and creative thinking.

3.3 To Q3

There were 75 male students (54,3%) and 63 female students (45,7%). Some interesting things in the comparison between males and females were drawn from Table 5: two male students did not have any solution ways, 16 was the number students with two solution ways and this number was equal in boys and girls. There were four female students with three correct solution ways who were superior to a male student. However, one student resolved the problem in four ways and this student was male.

In summary, the answer to Q3 was drawn as follows: In terms of problem solving, female students were somewhat better. However, in terms of the ability of creative thinking, male and female students were equal.

Table 5: Statistics of solutions of male and female students

The number of ways	Male Students	Female students
0	2	0
1	55	43
2	16	16
3	1	4
4	1	0

3.4 To Q4

3.4.1 Classical geometry method

In this method, S1.2 was chosen by a large number of students in three ways. S1.2 and S1.3 were the two solution ways in which students solved correctly. Particularly S1.1, students found it difficult to find the distance from the point *O* to the plane (*SCD*) with the following reasons:

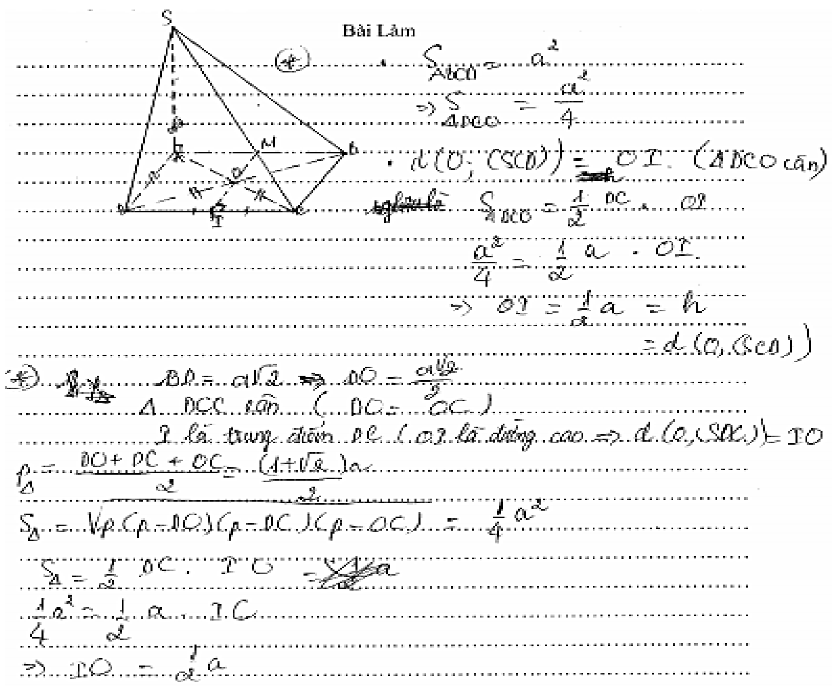

Fig. 6: A student's error related to the classical geometry method

– Students misunderstood the perpendicularity of straight lines and planes.

– Students determined the correct distance from *O* to plane *(SCD)*, but they did not yet complete, misleading the results because students misunderstood $SO \perp (ABCD)$, so they inferred that $\Delta SOA, \Delta SOM$ were right at *O*.

3.4.2 Volume method

– Students were not able to calculate the area for the following reasons:

\+ They did not miss Heron's formula. ($S=\sqrt{p(p-a)(p-b(p-c)}; p=\frac{a+b+c}{2}$).

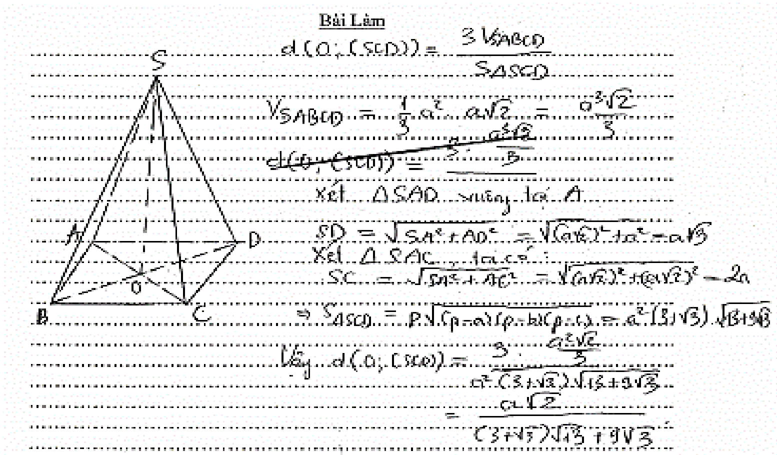

Fig. 7: A student's error related to missing Heron's formula

+ Students calculated *SD* wrongly.

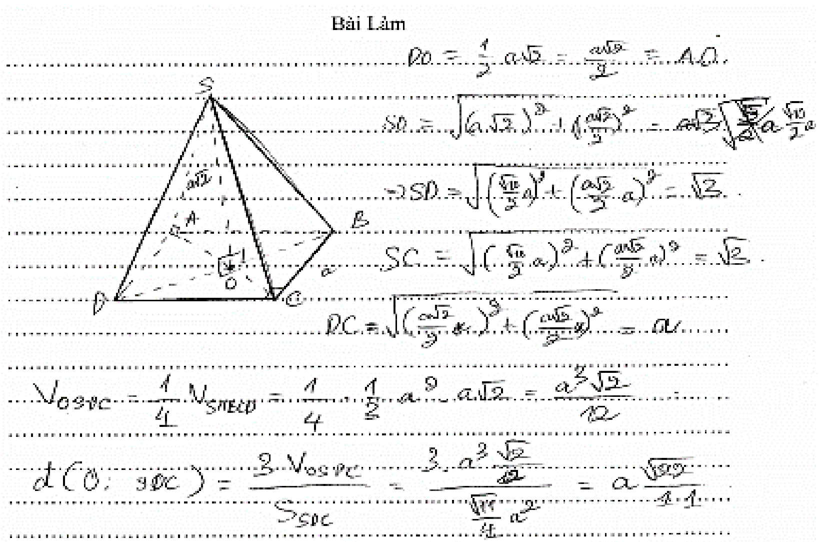

Fig. 8: A student's error related to calculating SD wrongly

– Students computed the volume incorrectly.

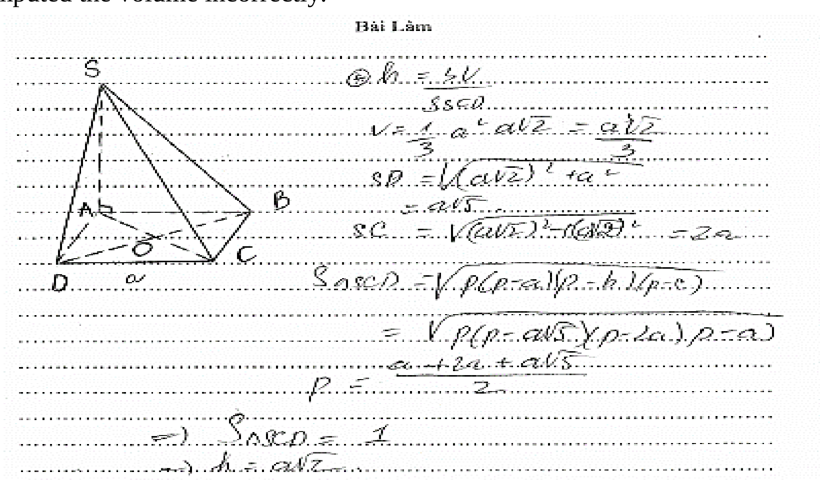

Fig. 9: A student's error related to calculating the volume wrongly

3.4.3 Coordinate method

Almost all of the students used this method and chose the correct origin of *Oxyz* coordinate system, but they still had the following errors:

– Wrong plane equation (*SCD*) led to the incorrect distance result, because they were wrong at the step of calculating the normal vector of the plane (*SCD*).

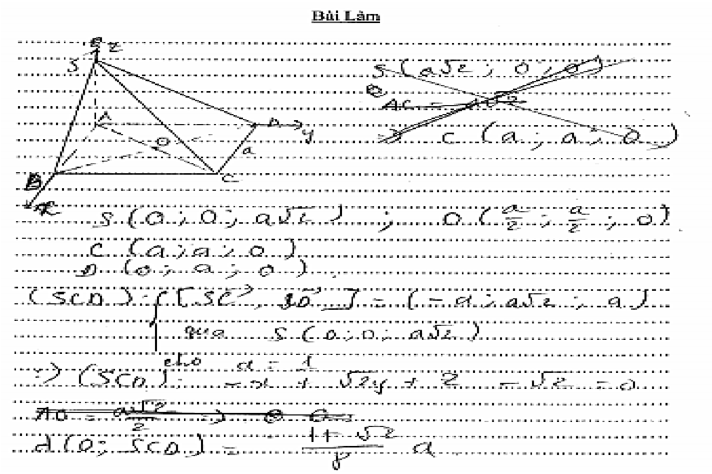

Fig. 10: A student's error related to calculating the normal vector of the plane wrongly

– They correctly applied the formula to calculate the distance from a point to a plane, but the end result was wrong, and the reasons were the students' miscalculation and carelessness.

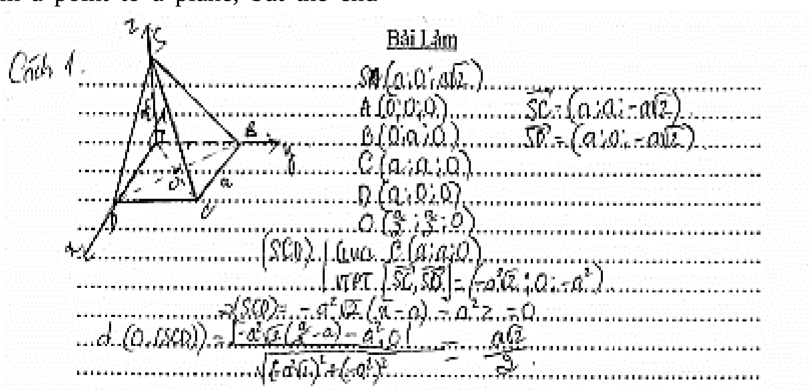

Fig. 11: A student's error related to miscalculation and carelessness

4 CONCLUSIONS

The results showed that the majority of students only solved the given problem in the way associated with the coordinate method. In other words, the preferred strategy was the coordinate method. In general, students at Can Tho were better at solving the problem in different ways than students in Hau Giang. Also, it was revealed that male students were not as good at solving the problem in various ways as female students. Besides, some students made the errors because they did not succeed in calcualting the distance based on the volume method and they forgot Heron's formula for calculating the area of a triangle. The classical geometric method was chosen by few students due to the fact that it was presented in the 11^{th} grade mathematics textbook, which led them to forget how to determine the projection of the point on the plane. In addition to the wrong calculation errors, students also made mistakes due to incorrect memory and carelessness. As a result, teachers need to acknowledge these errors so that they can adjust their teaching methods in order to help students realize the errors.

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